Observables in the Quantum Field Theory of neutrino mixing and oscillations

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Abstract

We report about recent results on the Quantum Field Theory of neutrino mixing and oscillations. A discussion of the relevant observables for flavor fields is given, leading to oscillation formulas which exhibit corrections with respect to the usual ones.

1 Introduction

Recent experimental results [1] have finally confirmed the reality of neutrino mixing and oscillations [2], after a long search. Despite these successes, many theoretical aspects of this problem are still unclear. In particular, difficulties arise already when attempting to find a proper mathematical setting for the description of mixing in the framework of Quantum Field Theory (QFT).

This is indeed an important task since it is well known [3] that mixing of states with different masses is not even allowed in non-relativistic Quantum Mechanics (QM). Nevertheless, the quantum mechanical treatment is the one usually adopted for its simplicity and elegance [2]. A review of the problems connected with the QM treatment of mixing and oscillations can be found in Ref.[4]. Difficulties in the construction of the Hilbert space for mixed neutrinos were pointed out in Ref.[5].

Only recently [6]-[18] a consistent treatment of mixing and oscillations[1] in QFT has been achieved and we report here on some of these developments.

2 Neutrino mixing in Quantum Field Theory

The quantization of mixed Dirac fields has been studied in detail in Refs.[6, 8, 10, 11, 14]. Here we report the main results for the case of two flavors. Let us consider the usual mixing relations connecting the flavor fields ν_e and ν_μ with the free fields ν_1 and ν_2 with definite masses m_1 and m_2:

$$\begin{aligned} \nu_e(x) &= \cos\theta\, \nu_1(x) + \sin\theta\, \nu_2(x) \\ \nu_\mu(x) &= -\sin\theta\, \nu_1(x) + \cos\theta\, \nu_2(x)\,, \end{aligned} \tag{1}$$

where θ is the mixing angle. We can write Eqs.(1) as

$$\nu_\sigma(x) \equiv G_\theta^{-1}(t)\, \nu_j(x)\, G_\theta(t), \tag{2}$$

with $(\sigma, j) = (e, 1), (\mu, 2)$ and where

$$G_\theta(t) = \exp\left[\theta \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right]\,. \tag{3}$$

[1]For a discussion of the existing approaches to neutrino oscillations in QFT, see Ref.[19].

The generator of mixing transformations has been studied in Ref.[6] where it was shown that its action on the vacuum $|0\rangle_{1,2}$ for the fields ν_j results in a new state $|0\rangle_{e,\mu}$ – the flavor vacuum:

$$|0(t)\rangle_{e,\mu} \;\equiv\; G_\theta^{-1}(t)\;|0\rangle_{1,2}\;, \tag{4}$$

which is orthogonal to $|0\rangle_{1,2}$ in the infinite volume limit. In the following, we will use $|0\rangle_{e,\mu} \equiv |0(0)\rangle_{e,\mu}$.
The free fields ν_j (j=1,2) can be quantized in the usual way (we use $t \equiv x_0$) [20]:

$$\nu_j(x) = \sum_{r=1,2} \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \left[u^r_{\mathbf{k},j}(t)\alpha^r_{\mathbf{k},j} \;+ v^r_{-\mathbf{k},j}(t)\beta^{r\dagger}_{-\mathbf{k},j}\right] e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad j = 1, 2\;, \tag{5}$$

with $u^r_{\mathbf{k},j}(t) = e^{-i\omega_{k,j}t}u^r_{\mathbf{k},j}$, $v^r_{\mathbf{k},j}(t) = e^{i\omega_{k,j}t}v^r_{\mathbf{k},j}$ and $\omega_{k,j} = \sqrt{\mathbf{k}^2 + m_j^2}$. The anticommutation relations are the usual ones; the wave function orthonormality and completeness relations are those of Ref.[6].

By use of $G_\theta(t)$, the flavor fields can be expanded as:

$$\nu_\sigma(x) \;=\; \sum_{r=1,2} \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \left[u^r_{\mathbf{k},j}(t)\alpha^r_{\mathbf{k},\sigma}(t) + v^r_{-\mathbf{k},j}(t)\beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\right] e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{6}$$

with $(\sigma, j) = (e, 1), (\mu, 2)$. The flavor annihilation operators are defined as $\alpha^r_{\mathbf{k},\sigma}(t) \equiv G^{-1}_\theta(t)\alpha^r_{\mathbf{k},j}G_\theta(t)$ etc. They clearly act as annihilators for the flavor vacuum Eq.(4). For further use, it is helpful to list them explicitly (see also Ref.[6]). In the reference frame with $\mathbf{k} = (0, 0, |\mathbf{k}|)$ the spins factorize and we have the simple expressions:

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\;\alpha^r_{\mathbf{k},1} \;+\; \sin\theta\;\left(U^*_{\mathbf{k}}(t)\;\alpha^r_{\mathbf{k},2} \;+\; \epsilon^r\;V_{\mathbf{k}}(t)\;\beta^{r\dagger}_{-\mathbf{k},2}\right)\,, \tag{7}$$

$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\;\alpha^r_{\mathbf{k},2} \;-\; \sin\theta\;\left(U_{\mathbf{k}}(t)\;\alpha^r_{\mathbf{k},1} \;-\; \epsilon^r\;V_{\mathbf{k}}(t)\;\beta^{r\dagger}_{-\mathbf{k},1}\right)\,, \tag{8}$$

$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\;\beta^r_{-\mathbf{k},1} \;+\; \sin\theta\;\left(U^*_{\mathbf{k}}(t)\;\beta^r_{-\mathbf{k},2} \;-\; \epsilon^r\;V_{\mathbf{k}}(t)\;\alpha^{r\dagger}_{\mathbf{k},2}\right)\,, \tag{9}$$

$$\beta^r_{-\mathbf{k},\mu}(t) = \cos\theta\;\beta^r_{-\mathbf{k},2} \;-\; \sin\theta\;\left(U_{\mathbf{k}}(t)\;\beta^r_{-\mathbf{k},1} \;+\; \epsilon^r\;V_{\mathbf{k}}(t)\;\alpha^{r\dagger}_{\mathbf{k},1}\right)\,, \tag{10}$$

where $\epsilon^r = (-1)^r$ and

$$U_{\mathbf{k}}(t) \equiv u^{r\dagger}_{\mathbf{k},2}(t)u^r_{\mathbf{k},1}(t) = v^{r\dagger}_{-\mathbf{k},1}(t)v^r_{-\mathbf{k},2}(t) \;=\; |U_{\mathbf{k}}|\;e^{i(\omega_{k,2}-\omega_{k,1})t}\,, \tag{11}$$

$$V_{\mathbf{k}}(t) \equiv \epsilon^r\;u^{r\dagger}_{\mathbf{k},1}(t)v^r_{-\mathbf{k},2}(t) = -\epsilon^r\;u^{r\dagger}_{\mathbf{k},2}(t)v^r_{-\mathbf{k},1}(t) \;=\; |V_{\mathbf{k}}|\;e^{i(\omega_{k,2}+\omega_{k,1})t}\,, \tag{12}$$

$$|U_{\mathbf{k}}| = \frac{|\mathbf{k}|^2 + (\omega_{k,1}+m_1)(\omega_{k,2}+m_2)}{2\sqrt{\omega_{k,1}\omega_{k,2}(\omega_{k,1}+m_1)(\omega_{k,2}+m_2)}} \quad ; \quad |V_{\mathbf{k}}| = \frac{(\omega_{k,1}+m_1) - (\omega_{k,2}+m_2)}{2\sqrt{\omega_{k,1}\omega_{k,2}(\omega_{k,1}+m_1)(\omega_{k,2}+m_2)}}\,|\mathbf{k}|\,. \tag{13}$$

In Eqs.(7)-(10) a rotation is combined with a Bogoliubov transformation, where the Bogoliubov coefficients satisfy $|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$. Similar results hold for Majorana and boson fields [7, 9, 16].

3 Observables for mixed neutrinos

In the previous Section we have seen how to quantize mixed fermion (neutrino) fields, leading to the expansion (6) for the fields with definite flavor. We have also seen that the vacuum structure is affected by the action of the mixing generator (3) which results in the flavor vacuum $|0\rangle_{e,\mu}$ and in the non-trivial structure of the flavor annihilation/creation operators (7)-(10).

The question now is to see what are the physical implications of these mathematical structures. To this end we address the question of what are the observable quantities for mixed fields. Let us

start with a discussion of the flavor states [8, 15, 16]. By definition, these states have definite flavor charge and so, for neutrino and antineutrino states (denoted as $|\nu_\sigma\rangle$ and $|\bar{\nu}_\sigma\rangle$) we should have[2]

$$Q_\sigma\,|\nu_\sigma\rangle \;=\; |\nu_\sigma\rangle \quad ; \qquad Q_\sigma\,|\bar{\nu}_\sigma\rangle \;=\; -|\bar{\nu}_\sigma\rangle\,. \tag{14}$$

The flavor charges for mixed fields have been studied in detail in Refs.[13, 7, 14]. In the present case of mixing of two Dirac fields, we obtain:

$$Q_\sigma(t) \;\equiv\; \int d^3\mathbf{x}\, \nu_\sigma^\dagger(x)\nu_\sigma(x) \;=\; \sum_r \int d^3\mathbf{k}\left(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\alpha^r_{\mathbf{k},\sigma}(t) \;-\; \beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\beta^r_{-\mathbf{k},\sigma}(t)\right)\,, \qquad \sigma = e,\mu, \tag{15}$$

where $Q_e(t) + Q_\mu(t) = Q$ with Q being the total (conserved) $U(1)$ charge [13]. Thus the flavor charges are diagonal in the flavor ladder operators. This is evident when we realize how they are related to the Noether charges[3] Q_j [13]:

$$Q_\sigma(t) \;=\; G^{-1}_\theta(t)\, Q_j\, G_\theta(t)\,, \qquad\qquad (\sigma, j) = (e,1), (\mu,2). \tag{16}$$

We thus are led to the following definition for a neutrino state with definite flavor:

$$|\nu_\sigma\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},\sigma}(0)|0\rangle_{e,\mu} = G^{-1}_\theta(0)\alpha^{r\dagger}_{\mathbf{k},j}|0\rangle_{1,2}\ . \tag{17}$$

with similar expressions for antineutrinos. Clearly, the state (17) satisfies the requirement of Eq.(14). Moreover, we can see that $|\nu_\sigma\rangle$ also satisfies

$$\mathbf{P}_\sigma(0)\,|\nu_\sigma\rangle \;=\; \mathbf{k}\,|\nu_\sigma\rangle \tag{18}$$

where the momentum operator for the mixed fields is defined as [16] ($\sigma = e,\mu$):

$$\begin{aligned}\mathbf{P}_\sigma(t) \;&=\; \int d^3\mathbf{x}\, \nu_\sigma^\dagger(x)\,(-i\nabla)\,\nu_\sigma(x) \;=\; G^{-1}_\theta(t)\, P_j\, G_\theta(t)\\ &=\; \int d^3\mathbf{k} \sum_r \frac{\mathbf{k}}{2}\left(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\alpha^r_{\mathbf{k},\sigma}(t) - \alpha^{r\dagger}_{-\mathbf{k},\sigma}(t)\alpha^r_{-\mathbf{k},\sigma}(t) + \beta^{r\dagger}_{\mathbf{k},\sigma}(t)\beta^r_{\mathbf{k},\sigma}(t) - \beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\beta^r_{-\mathbf{k},\sigma}(t)\right).\end{aligned} \tag{19}$$

Note that the usually employed Pontecorvo states $|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$ and $|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$ are not eigenstates of the flavor charge neither of the momentum operator (see also Ref.[21]) and thus are not consistently defined within QFT.

4 Oscillation formulas

We now use the above results to derive oscillation formulas. Let us consider the case of an electron neutrino state $|\nu_e\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},e}|0\rangle_{e,\mu}$. At time $t \neq 0$, this is not anymore eigenstate of the flavor charge operators. We obtain ${}_{e,\mu}\langle 0|Q_\sigma(t)|0\rangle_{e,\mu} \;=\; 0$ and

$$\mathcal{Q}_{\mathbf{k},\sigma}(t) \;\equiv\; \langle \nu_e|Q_\sigma(t)|\nu_e\rangle \;=\; \left|\left\{\alpha^r_{\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 \;+\; \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2\,. \tag{20}$$

Charge conservation is obviously ensured at any time: $\mathcal{Q}_{\mathbf{k},e}(t) + \mathcal{Q}_{\mathbf{k},\mu}(t) \;=\; 1$. The oscillation formulas for the flavor charges are [8]

$$\mathcal{Q}_{\mathbf{k},e}(t) \;=\; 1 \;-\; \sin^2(2\theta)\,|U_\mathbf{k}|^2\ \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right) + \sin^2(2\theta)\,|V_\mathbf{k}|^2\ \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right)\,, \tag{21}$$

$$\mathcal{Q}_{\mathbf{k},\mu}(t) \;=\; \sin^2(2\theta)\,|U_\mathbf{k}|^2\ \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right) + \sin^2(2\theta)\,|V_\mathbf{k}|^2\ \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right)\,. \tag{22}$$

[2]In the following, we work in the Heisenberg picture.

[3]These are the $U(1)$ charges separately conserved for the free fields ν_j. We have: $Q_1 + Q_2 = Q$.

This result is exact. The difference with respect to the Pontecorvo formula [2] is in the energy dependent amplitudes and in the additional oscillating terms. The usual QM formulas [2], are approximately recovered in the relativistic limit ($k \gg \sqrt{m_1 m_2}$) where we obtain (for $\theta = \pi/4$):

$$
\mathcal{Q}_{\mathbf{k},\mu}(t) \simeq \left(1 - \frac{(\Delta m)^2}{4k^2}\right) \sin^2\left[\frac{\Delta m^2}{4k}\, t\right] + \frac{(\Delta m)^2}{4k^2} \sin^2\left[\left(k + \frac{m_1^2 + m_2^2}{4k}\right) t\right] . \tag{23}
$$

Similar results are obtained when we consider the expectation values of the momentum operator at a time $t \neq 0$. We have indeed ${}_{e,\mu}\langle 0|\mathbf{P}_\sigma(t)|0\rangle_{e,\mu} = 0$ and

$$
\frac{\langle \nu_e|\mathbf{P}_\sigma(t)|\nu_e\rangle}{\langle \nu_e|\mathbf{P}_\sigma(0)|\nu_e\rangle} = \left|\left\{\alpha^r_{\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 , \tag{24}
$$

which is the same expression obtained for the charges Eq.(20). Note that the momentum operator is well defined for Majorana fields, whereas the flavor charge operators vanish for neutral fields [16].

5 Discussion and conclusions

We have seen how the fields ν_e and ν_μ can be expanded in the same spinor bases as ν_1 and ν_2, viz. Eq.(6). However, such a choice is actually a special one and a more general possibility exists [10]. Indeed, in the expansion (6) one could use eigenfunctions with arbitrary masses μ_σ and write the flavor fields as [10]:

$$
\nu_\sigma(x) = \sum_{r=1,2} \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \left[u^r_{\mathbf{k},\sigma} \widetilde{\alpha}^r_{\mathbf{k},\sigma}(t) + v^r_{-\mathbf{k},\sigma} \widetilde{\beta}^{r\dagger}_{-\mathbf{k},\sigma}(t)\right] e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{25}
$$

where u_σ and v_σ are the eigenfunctions with mass μ_σ. We denote by a tilde the generalized flavor operators introduced in Ref.[10]. The expansion Eq.(25) is more general than the one in Eq.(6) since the latter corresponds to the particular choice $\mu_e \equiv m_1$, $\mu_\mu \equiv m_2$. Thus the Hilbert space for the flavor fields is not unique: an infinite number of vacua can be generated by introducing the arbitrary mass parameters μ_σ. It is obvious that physical quantities must not depend on these parameters. Similar results are valid for bosons, see Ref.[7]. It can be explicitly checked that ($\sigma, \rho = e, \mu$) [12]:

$$
\left|\left\{\widetilde{\alpha}^r_{\mathbf{k},\sigma}(t), \widetilde{\alpha}^{r\dagger}_{\mathbf{k},\rho}(t')\right\}\right|^2 + \left|\left\{\widetilde{\beta}^{r\dagger}_{-\mathbf{k},\sigma}(t), \widetilde{\alpha}^{r\dagger}_{\mathbf{k},\rho}(t')\right\}\right|^2 = \left|\left\{\alpha^r_{\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},\rho}(t')\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},\rho}(t')\right\}\right|^2 , \tag{26}
$$

which ensures the cancellation of the arbitrary mass parameters in the expectation values (20),(24). Thus the independence of expectation values on the arbitrary parameters provides a criterion for the selection of the observables for mixed fields. Indeed, the number operators for mixed fields are not good observables since their expectation values do depend on the arbitrary mass parameters.

In conclusion, we have discussed in this report how to properly define observable quantities for mixed (Dirac) fields in the context of Quantum Field Theory. We derived oscillation formulas which exhibit corrections with respect to the usual ones.

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References

[1] J. Davis, D. S. Harmer and K. C. Hoffmann, Phys. Rev. Lett. **20** (1968) 1205; M. Koshiba, in "Erice 1998, From the Planck length to the Hubble radius" 170; S. Fukuda et al. (Super-Kamiokande collaboration), Phys. Rev. Lett. **86** (2001) 5656; Q. R. Ahmad et al. (SNO collaboration) Phys. Rev. Lett. **87** (2001) 071301; Phys. Rev. Lett. **89** (2002) 011301; K. Eguchi *et al.* [KamLAND Collaboration], Phys. Rev. Lett. **90** (2003) 021802; M. H. Ahn *et al.* [K2K Collaboration], Phys. Rev. Lett. **90** (2003) 041801.

[2] B. Pontecorvo, Zh. Eksp. Theor. Fiz. **33** (1958) 549; JEPT **6** (1958) 429; Z. Maki, M. Nakagawa and S. Sakata, Prog. Theor. Phys. **28** (1962) 870; V. Gribov and B. Pontecorvo, Phys. Lett. **B28** (1969) 493; S. M. Bilenky and B. Pontecorvo, Phys. Rep. **41** (1978) 225.

[3] V. Bargmann, Annals Math. **59** (1954) 1; A. Galindo and P. Pascual, *Quantum Mechanics*, (Springer, 1990). See also: D. M. Greenberger, Phys. Rev. Lett. (2001).

[4] M. Zralek, Acta Phys. Polon. **B29** (1998) 3925.

[5] C. Giunti, C. W. Kim and U. W. Lee, Phys. Rev. **D45** (1992) 2414; C.W. Kim and A. Pevsner, *Neutrinos in Physics and Astrophysics*, Harwood Academic Press, 1993.

[6] M. Blasone and G. Vitiello, Annals Phys. **244** (1995) 283 [Erratum-ibid. **249** (1995) 363].

[7] M. Blasone, P. A. Henning and G. Vitiello, [hep-ph/9605335]. M. Blasone, A. Capolupo, O. Romei and G. Vitiello, Phys. Rev. **D63** (2001) 125015.

[8] M. Blasone, P. A. Henning and G. Vitiello, Phys. Lett. **B451** (1999) 140; M. Blasone, in "Erice 1998, From the Planck length to the Hubble radius" 584, [hep-ph/9810329].

[9] M. Binger and C. R. Ji, Phys. Rev. **D60** (1999) 056005. C. R. Ji and Y. Mishchenko, Phys. Rev. D 64 (2001) 076004. Phys. Rev. **D65** (2002) 096015.

[10] K. Fujii, C. Habe and T. Yabuki, Phys. Rev. **D59** (1999) 113003 Phys. Rev. **D64** (2001) 013011.

[11] K. C. Hannabuss and D. C. Latimer, J. Phys. **A36** (2003) L69; J. Phys. **A33** (2000) 1369.

[12] M. Blasone and G. Vitiello, Phys. Rev. **D60** (1999) 111302.

[13] M. Blasone, P. Jizba and G. Vitiello, Phys. Lett. **B517** (2001) 471.

[14] M. Blasone, A. Capolupo and G. Vitiello, Phys. Rev. **D66** (2002) 025033;

[15] M. Blasone, P. P. Pacheco and H. W. Tseung, Phys. Rev. **D67** (2003) 073011.

[16] M. Blasone and J. S. Palmer, [hep-ph/0305257]

[17] K. Fujii, C. Habe and M. Blasone, [hep-ph/0212076].

[18] M. Blasone, J. Magueijo and P. Pires-Pacheco, [hep-ph/0307205].

[19] M. Beuthe, Phys. Rep. **375** (2003) 105.

[20] H. Umezawa, *Advanced Field Theory: Micro, Macro and Thermal Physics* (AIP, 1993)

[21] C. Giunti, Mod. Phys. Lett. A **16** (2001) 2363.